COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 9 (2) 1989 227—229

NOTE

AHYPERGRAPH-FREE CONSTRUCTION OF HIGHLY CHROMATIC GRAPHS WITHOUT SHORT CYCLES

IGOR KŘÍŽ

Received October 9, 1986

We present a purely graph-theoretical construction of highly chromatic graphs without short cycles.

The first known construction of a graph $G(n, i)$ of chromatic number at least n without cycles of length $<i$ was given in [1]. Another, substantially simpler one has been presented in [4]. Both of theese constructions are based on an induction, solving the problem for graphs and hypergraphs simultaneously. From the graph-theoretical point of view, a graph constructed in this way, involving a number of hypergraph-steps, becomes rather obscure. The best graph-theoretical construction is presented in [2] and if deals with a special case only. The problem of finding a general graphical construction is well known (see e.g. [5], [3]).

In this note, we present a purely graph-theoretical construction of a $G(n, i)$. In some sense, our $G(n, i)-s$ are really simpler than those of [1] and [4]. In particular, we can construct them as a union of $n-1$ bushes (forests with components of diameter $\leqq 2$). From other points of view, however, our construction is still far from satisfactory: for instance, as in the previous constructions, $G(n, i)$ is still not primitively recursive in i.

In the sequel, a circle in a *labeled graph* is a graph with edges labeled by natural numbers. The length of a labeled graph is defined as the sum of the numbers assigned to its edges. The edges of an ordinary graph are automatically considered to be labeled by one. The complete graph on a set X with all edges labeled by k will be denoted by $K^k(X)$. A *bush* is a forest with components of diameter $\leqq 2$. For a graph G $V(G)$ and $E(G)$ will be its set of vertices and edges, respectively. As usual, an *independent set* of a graph is a set of vertices containing no edge.

Lemma. *There exists a bush $\Gamma_i(n)$ with the following properties:*

(1) *We have $V(\Gamma_i(n)) = \bigcup \mathscr{C}_i^n$ where $\mathscr{C}_i^n$ is a system of disjoint independent sets of cardinality n.*

(2) *The labeled graph $\Gamma_i^k(n) = \Gamma_i(n) \cup \bigcup_{X \in \mathscr{C}_i^n} K^k(X)$ has no cycles of length $\leqq 2^i(k+1) + 2^{i+1} + 2$ except, possibly, the cycles in $\bigcup_{X \in \mathscr{C}_i^n} K^k(X)$.*

(3) *If M is an independent set in $\Gamma_i(n)$, then there exists an $X \in \mathscr{C}_i^n$ with $X \cap M = \emptyset$.*

AMS subject classification (1980): 05 C 15

Proof. *Construction:* Define $\Gamma_i(n)$, $i \geqq 0$, $n \geqq 1$ inductively by

(a) $$V(\Gamma_i(1)) := \{0, 1\},\ E(\Gamma_i(1)) := \{\{0, 1\}\},\ \mathscr{C}_i^1 := \{\{0\}, \{1\}\}.$$

(b) $$V(\Gamma_0(n)) := \{0, \dots, n-1\} \cup (\{0, \dots, n-1\} \times \{0, \dots, n-1\}),$$

$$E(\Gamma_0(n)) := \{\{i, (i, j)\} | i, j \in \{0, \dots, n-1\}\}$$

$$\mathscr{C}_0^n := \{\{0, \dots, n-1\}\} \cup$$

$$\cup \{\{i\} \times \{0, \dots, n-1\} | i \in \{0, \dots, n-1\}\}.$$

(c) For $i > 0$, $n > 1$ put $\overline{\mathscr{C}_i^n} := \mathscr{C}_{i-1}^{|\mathscr{C}_i^{n-1}|}$ and choose a bijection $\psi_X : X \cong \mathscr{C}_i^{n-1} \times \{X\}$ for each $X \in \overline{\mathscr{C}_i^n}$. Moreover, let, for a graph G, G_X be defined by

$$V(G_X) := V(G) \times \{X\},$$

$$E(G_X) := \{\{(a, X), (b, X)\} | \{a, b\} \in E(G)\}.$$

Then define

$$\Gamma_i(n) := \Gamma_{i-1}(|\mathscr{C}_i^{n-1}|) \cup \bigcup_{X \in \overline{\mathscr{C}_i^n}} \Gamma_i(n-1)_X,$$

$$\mathscr{C}_i^n := \{\{z\} \cup \Psi_X(z) | z \in X \in \overline{\mathscr{C}_i^n}\}.$$

Now (1) is obvious, as is the fact that $\Gamma_i(n)$ is a bush. The proofs of (2), (3) will be given by induction on (i, n).

Proof of (3): (a) and (b) are obvious. (c) Let M be an independent set in $\Gamma_i(n)$. By the induction hypothesis we have an $X \in \overline{\mathscr{C}_i^n}$ with $M \cap X = \emptyset$. Similarly, there exists a $Y \in \mathscr{C}_i^{n-1} \times \{X\}$ such that $M \cap Y = \emptyset$. We conclude that $M \cap (Y \cup \{\psi_X^{-1}(Y)\}) = \emptyset$.

Proof of (2): (a) and (b) are obvious. (c) Put $\Gamma := \Gamma_{i-1}(|\mathscr{C}_i^{n-1}|) \subset \Gamma_i^k(n)$. Evidently, the only pairs of vertices of Γ joined by path in $\Gamma_i^k(n) \setminus \Gamma$ are of the form $y \neq z \in X$ for some $X \in \mathscr{C}_i^n$ Moreover, any such path contains at least two edges in $\bigcup_X K^k(X)$ and at least one edge from $\Gamma_i(n)$, which implies that is length is at least $2k+1$. Thus, any cycle in $\Gamma_i^k(n)$, containing at least one edge from Γ, is at least as long as a cylce in $\Gamma_{i-1}^{2k+1}(|\mathscr{C}_i^{n-1}|)$, i.e., at least $1 + 2^{i-1}(2k+1+1) = 1 + 2^i(k+1)$ long. On the other hand, any cycle from $\Gamma_i^k(n)$, containing no edge from Γ and not contained in $\bigcup_X K^k(X)$ is at least as long as a cycle in $\Gamma_i^k(n-1)_X$ with an $X \in \mathscr{C}$. ∎

Let $G = (V, E)$ be a graph. Choose bijections $\varphi_X : V \to X$ for $X \in \mathscr{C}_i^{|V|}$ and put

$$\Gamma_i(G) := \Gamma_i(V) \cup \bigcup_{X \in \mathscr{C}_i^{|V|}} \varphi_X^*(G),$$

where φ_X^* is the induced map (i.e. $\varphi_Y^*(G)$ is an isomorphic copy of G on the vertex set X).

Theorem. *The graph*

$$\underbrace{\Gamma_i(\Gamma_i \ldots \Gamma_i}_{n \text{ times}}(1))$$

has chromatic number at least $n+1$ *and has no cycles of length* $\leqq 2^{i+1}$. *Moreover, it is a union of n bushes.*

Proof is immediately deduced from Lemma: the bound on the chromatic number is deduced from (3) and the cycle length is deduced from (2). The decomposition into bushes an easy induction on n. ∎

Acknowledgement. I am very indebted to J. Nešctřil for discussions on the topic.

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I. Kříž

The Ohio State University
Department of Mathematics
Columbus, Ohio 43210
U.S.A.